If the boundary condition is

$$\mathrm{V} = 0 \text{ when } x = r(\theta)\cos\theta, \qquad y = r(\theta)\sin\theta,$$

we obtain the integral equation

$$0 = \int_0^{2\pi} e^{ikr(\theta)\cos(\theta-\alpha)}\,\phi(\alpha)\,d\alpha,$$

which may be regarded as a limiting case of the homogeneous equation

$$\mu(k)\,\phi(\theta) = \int_0^{2\pi} e^{ikr(\theta)\cos(\theta-\alpha)}\,\phi(\alpha)\,d\alpha.$$

We may apply our method of approximation to obtain the possible values of the functions $\mu(k)$. Equating these to zero we obtain the possible values of k.

The Atomic Process in Ferromagnetic Induction.

By Sir J. Alfred Ewing, K.C.B., F.R.S.

(Received November 29, 1921.)

The object of this paper is to amend, in an important particular, the theory of ferromagnetic induction put forward by me more than 30 years ago,* and to describe a new model. That theory was itself a modification of the earlier theory of Weber. To Weber is due the fundamental notion that a substance contains minute particles, each of which acts as a magnet, and that in the process of magnetising a ferromagnetic substance these are turned into more or less complete alignment. The ultimate magnetic particles used to be called "molecular magnets": we now recognise them as attributes of the atom, not of the molecule, and (in all probability) they derive their magnetic moment from the circulation of electricity in electron orbits or in ring electrons. What turns is not the molecule nor the atom, but something within the atom.

The characteristics which distinguish ferromagnetic substances from other paramagnetics are: (1) the much larger amount of magnetism they can acquire under the action of an impressed field; (2) the fact that the acquired magnetism tends towards a saturation limit when the field is progressively increased; (3) the fact that the acquired magnetism shows hysteresis with respect to variations of the field, except in certain small initial changes. Weber's theory explained (1) and (2). My modification of it explained, in addition, (3) as an effect of the irreversible action which occurs when the equilibrium of a magnetic element becomes unstable through change in the externally impressed magnetic force, and it swings over, with dissipation of energy, into a new position of stability. The stability in both positions is sufficiently explained by magnetic forces only. In breaking away from one stable position it is deflected at first in a quasi-elastic (reversible) manner until the external force reaches a certain value at which the equilibrium is upset. The essence of hysteresis is the turning from one position of stability to another, through a region of instability. If the conditions are such that there is no unstable phase in the turning, then there is no dissipation of energy, and consequently no hysteresis. This occurs in very feeble magnetisation, when the deflections are reversible; it also occurs if the piece be caused to rotate in a field of great strength. J. Swinburne pointed out that, as a consequence of my theory, hysteresis should vanish when a cylinder of ferromagnetic metal is rotated in a very strong field,* and this curious result was confirmed experimentally by F. G. Baily.†

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* "Contributions to the Molecular Theory of Induced Magnetism," 'Roy. Soc. Proc.' vol. 48, p. 342 (1890); 'Phil. Mag.,' September, 1890. See also 'Magnetic Induction in Iron and other Metals,' Chap. XI.

So far the theory is, I think, still valid. What needs amendment is that the magnetic control of each Weber element was ascribed simply to the action of its neighbours, namely, other Weber elements which were also liable to be turned by a sufficient impressed force. The Weber elements were treated as deriving their stability by forming rows or pairs, in which the mutual magnetic actions between similar neighbouring elements in the row constituted the controlling force. Hysteresis was found when rows (or pairs) broke up and new rows (or pairs) were formed. A model was constructed to illustrate this action, with groups of pivoted permanent magnets, and it was shown to reproduce with remarkable completeness, in a qualitative manner, the chief features of magnetisation as observed in iron and other ferromagnetic substances.

One of these features is that there is no more than a very small reversible change of magnetism, before hysteresis asserts itself, when a field begins to be applied, or begins to be removed, or to be re-applied. Thus in the magnetisation of common soft iron from a neutral state the initial stage, which was shown by the late Lord Rayleigh‡ to be reversible, passes into an irreversible stage, involving hysteresis, when only a very small fraction of the magnetism of saturation has been acquired, say about 10 or 15 units out of 1700. This implies that the average amount of reversible deflection on the part of the Weber element before it becomes unstable is less than 1° In order to reproduce this feature in a model with pivoted magnets, we have to space the magnets so closely that when they group themselves in rows the poles are very near together. From calculations relating to the stability of a row of pivoted magnets, which I have checked experimentally by observing the behaviour of pairs and rows of ball-ended magnets in the field of a pair of coils arranged as in the Helmholtz galvanometer, I find that if the limiting stable deflection is to be of the order of 1°, the ratio a/r, where a is the half distance between centres and r is the half length of each magnet, must not exceed about 1·025.

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* 'Industries,' September 19, 1890.

† 'Phil. Trans.,' A, vol. 187, p. 715 (1896).

‡ 'Phil. Mag.,' March, 1887.

On the assumption that each atom in a crystal of iron contains one Weber element, it is easy to calculate what would be the strength of the field which would rupture a row, with given orientation, if the poles of the elements are spaced as closely as this condition demands. When the calculation is made it gives a field that is immensely greater than the field actually required, in iron, to pass from the reversible to the irreversible stage. The calculated field, for such rows (assuming the usual laws of force to apply) is some thousands of times greater than that at which hysteresis begins to show itself.* I have therefore been obliged to abandon the idea of representing the process of magnetisation of iron, even roughly, by means of magnetic elements all free to turn and controlling one another from atom to atom in the crystal. But another type of model has now suggested itself which seems to offer a satisfactory solution.

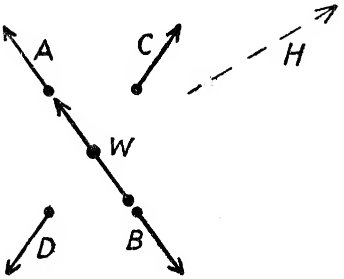

Fig I.

Imagine a pivoted magnet W (fig. 1), acted upon by a deflecting field H, and controlled by two *fixed* magnets A and B, which point in opposite directions, so that the control depends on the difference of their effects. Imagine a slight lack of symmetry, a difference either in strength or in nearness, so that the fixed magnet A acts rather more strongly than the fixed magnet B. Then W is in a feebly stable condition, which a weak deflecting field will suffice to disturb; and *provided the poles of A and B are near those of W*, there will be no more than a small reversible deflection before the equilibrium is upset. This group accordingly satisfies the two necessary conditions which were incompatible in a pair or row of pivoted magnets; it combines small limiting stable deflection with easy rupture of stability. Assume now that lying along a transverse line more nearly in the direction of H, there are two other fixed magnets C and D, also slightly unequal in their effects. When W breaks away from A and B it may swing (with dissipation of energy) into a stable relation with C and D, until finally when H is sufficiently increased it will be pulled out of that into nearly perfect parallelism with H. It will be obvious that such a model—when imagined as extended into three dimensions—reproduces ferromagnetic phenomena. In applying magnetising force to a piece of metal made up of groups of such atoms we should clearly find a slow beginning, a subsequent rapid rise, and a final slow approach to saturation; and in removing the force we should find residual magnetism.

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* In fairly pure soft iron the field that is sufficient to reach the limit of elastic magnetisation is about 1 C.G.S. unit, and in the purer iron that is obtained by electrolytic deposition, with subsequent melting *in vacuo*, it may be only about 0·1 C.G.S. unit. (See the experiments of Yensen, 'Illinois University Bulletins,' Nos. 72, 83, 95.)

I have made a one-plane model with ball-ended magnets, each consisting of a strongly magnetised straight steel wire with its ends screwed into quarter-inch steel balls. The central magnet W is pivoted like a compass needle. Each of the controlling magnets, A, B, C, D, is held in a fixed support, but so that it can be moved towards or away from the pivoted magnet, thus allowing the control to be made as weak or as strong as may be desired. When the group is set in a field which is increased, reduced, reversed and so on, or when the group is caused to turn in a constant field, the features observed in cyclic and other processes of magnetisation are very exactly imitated. The model may be made more realistic by including a number of similar contiguous groups, but a single group suffices to illustrate the action.

It remains to be seen how the parts which make up the model can be thought of as having equivalents in the structure of the atom. One has to imagine the electrons of each atom as furnishing two types of magnetic element, a turning element within the atom, and other elements at a greater distance from the centre, which are held fixed (more or less completely) as a consequence of the relation between the atom and the neighbouring atoms of the crystal. Take, for instance, a structure such as A. W. Hull has suggested, on the basis of his X-ray analysis, as a probable structure for the iron atom,* in which two of the electrons, on opposite sides of the nucleus, form a doublet, and the remaining twenty-four are grouped in octets, further from the nucleus, along the diagonals of a cube. Think provisionally of these octets as making up fixed magnetic elements which correspond to A, B, C, and D, of the model, with axes that lie in the cube diagonals of the atom, coinciding (it may be supposed) with the trigonal axes of the crystal. They point radially inwards, and all their inner "poles" are of the same name. In the inner space the doublet, which corresponds to W, the Weber element of the model, turns under the influence of any impressed field. When there is no external field, the Weber element may be in any one of eight positions of stability, pointing towards one of the fixed elements, namely towards one of the eight corners of a cube.* Its irreversible passage from one position of stability to another is what constitutes hysteresis. Its small reversible deflection, before breaking away from any one position of stability, constitutes the initial quasi-elastic stage of magnetisation. The range of this deflection is small because the controlling poles are near it. The stability is feeble because the control is differential: it depends on the nearly balanced action of large opposing forces which are separately applied by two fixed elements.

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* "The Crystal Structure of Iron," 'Phys. Rev.,' vol. 9, p. 85 (1917).

We have to postulate a small defect of symmetry in their action, in order that the Weber element may be stable. In any position which it may assume, the circumferential magnetic elements must act on its two ends somewhat unequally. The attraction in the model between one end and the unlike pole near it must be slightly stronger than the repulsion between the other end and the like pole near that. More than one cause may be operative in producing a corresponding asymmetry in the atom. The circumferential magnetic elements may not be strictly fixed, relative to the centre of rotation of the Weber element, but may be capable of small displacements of a kind to cause the unlike pole of one to act more strongly and the like pole of the other to act less strongly, or the Weber element may admit of being pulled a little nearer the attracting pole. It is at least possible that such displacements may result from the mutual magnetic forces between the Weber element and the fixed elements. The model with ball-ended steel magnets behaves, in fact, in a way that illustrates just such an action, for the magnetic rigidity of the bars is imperfect, and when the distances between the balls are adjusted to be equal at both ends the pivoted magnet is found to be stable, because magnetic induction has slightly altered the distribution of magnetism of the bars, bringing unlike poles nearer together, and setting like poles further apart.

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* According to Hull's X-ray analysis, the space-lattice of an iron crystal is the centred cube: the spacing of the atoms is closest along each trigonal axis. In the model, as shown in the figure, the plane of the sketch is that of two trigonal axes. Hull finds that the side of the unit cube—namely, the distance between atom centres along the cubic axes of the crystal—measures $2{\cdot}86 \times 10^{-8}$ cm. Since the space-lattice is the centred cube, there are two atoms in each unit cube, and $\frac{2}{(2{\cdot}86 \times 10^{-8})^3}$ atoms in a cubic centimetre. We may take the magnetism of saturation of iron to be 1700 C.G.S. units. Hence, assuming that there is one rotatable Weber element in each atom its moment is $\frac{1700\,(2{\cdot}86 \times 10^{-8})^3}{2}$ or 2×10^{-20} very nearly.

Further, some small geometrical asymmetry may be set up by the action of neighbouring atoms. Effects of strain on magnetic quality—which are known to be large and complex—may be explainable by reference to displacements of the external electrons and consequent disturbances of the balance on which the control of the Weber element depends. Even in unstrained iron an impurity may, in general, be expected to restrain the response to an impressed field by disturbing this balance. When an atom of foreign matter finds a place in the space-lattice the symmetry of the neighbours must be affected by its presence, the grouping of their electrons will be more or less distorted and the Weber elements in them may consequently be unduly stabilised. This is borne out by the observed fact that the hysteresis of iron becomes in general less and less, and the susceptibility becomes greater, the more completely all impurities, including dissolved gases, are eliminated. Again, in a chemical compound of iron the iron atom is unsymmetrically surrounded and its magnetic properties become widely different from those of an atom of uncombined iron. In these respects, and in others such as the effects of vibration and temperature, this new model seems to give results that agree with observation.

Objection may be taken that the Weber element of the model is too much buried within the shells of outer electrons to be susceptible to the influence of stress and vibration, or to the presence of another chemical constituent in the crystal. But it should be noted that according to the view I have endeavoured to set out, its stability depends on a delicate balance between forces which are exerted on it by the outer electrons, and will be readily affected by anything that affects them. That only an inner and probably small constituent of the atom should be liable to turn, in the process of magnetisation, agrees with the observations of A. H. Compton and O. Rognley, who could detect no change in the X-ray spectrum reflected from the face of an iron crystal when the crystal was magnetised.*

Their conclusion that the atom itself does not turn may be deduced without X-ray analysis from the simple consideration that, if the atom turned, we should expect the cohesion and rigidity of the piece to be much affected during a process such as the reversal of a strong external field, and of that there is no evidence. Everything points to the idea that in each atom of the crystal a shell of outer electrons is locked in relation to its neighbours, except for small relative movements such as occur through thermal agitation or mechanical straining, and that the Weber elementary magnet is constituted by other electrons nearer the nucleus.

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* 'Phys. Rev.,' vol. 16, p. 464 (1920). From earlier experiments on the dispersion of X-rays, K. T. Compton and Trousdale had inferred that the Weber element is not a molecule or group of atoms.—'Phys. Rev.,' vol. 5, p. 315 (1915).

We are not here concerned with the nature of the forces between neighbouring atoms in the space-lattice. If they are largely electromagnetic the contiguous electron orbits in any two nearest neighbours may be oppositely directed. In a model of iron, for instance, we might imagine one atom to have the equivalent of eight north poles on its surface, while each of its eight nearest neighbours has eight south poles on its surface. The space-lattice being the centred cube, each atom, under these conditions, would be surrounded by eight nearest neighbours of opposite polarity to itself. Such a grouping would obviously lend itself well to the building up of a model crystal in which the magnetic property of the electrons was represented by little bar magnets, facing radially, but in opposite senses in the shells of any two neighbouring atoms. So far as the action of the inner element is concerned—the element which turns in response to an applied field—it is indifferent in which sense the controlling magnets face, provided they all face in the same sense in any one atom.

A model satisfying the essential conditions may take more than one form. The form that has been described is appropriate if the electrons have orbits whose dimensions are small relatively to the size of the atom, or are themselves small rings. The conditions are simply that one or more of the orbits or rings, with an unbalanced magnetic moment, should be capable of orientation under control exerted by other parts of the atom (which are held relatively fixed), and that it should be deflected through only a small angle before becoming unstable, when it turns into another position of stability; and further that the stability in each position should be weak. The combination of weak stability with a narrow range of stable deflection is secured by the differential character of the control: it depends on there being a slight inequality between two strong but oppositely directed forces. These features are exemplified in the model already described (fig. 1). Another model, which also exemplifies them, but in a manner more suggestive of large orbits, is illustrated in fig. 2. There W (which represents the Weber element) is a coil carrying a current and free to turn about an axis *ab*. A and B are fixed controlling coils in close proximity, with currents as shown by the arrows, arranged to exert slightly unequal forces on W. By adjusting the distances, or the strength of the currents, the resulting control can be made as small as is desired. In the model there are two more controlling coils set in a plane transverse to that of the first pair, so that W may turn from one stable position to another (see plan, fig. 2*a*). In turning under the influence of an impressed field, it is deflected stably through a narrow range before becoming unstable.

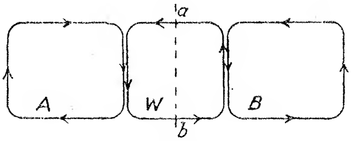

Fig 2

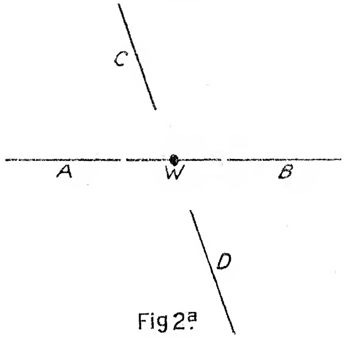

Fig 2a

It will be noted that in this model the Weber element is constituted by a single electron orbit, instead of being a doublet as in fig. 1. But the conditions governing the stability are in essential points the same; it still depends on the difference between two nearly equal forces, and this allows the control to be weak although the range of stability is kept small.

An interesting variant of this model, and one that perhaps comes nearer the real structure of the atom, is obtained by making the fixed coils A and B overlap sufficiently to embrace W, as in fig. 3. There A and B represent coplanar elliptic orbits with a common focus which is also the centre of a circular orbit, W, representing the Weber element. In the model the fixed controlling coils A and B are elliptical in form and are set as nearly as may be in one plane. W is a circular coil pivoted about the axis, *ab*. As before, the model is to be understood as having a second pair of similar controlling coils set in a transverse plane; and it is easy (in imagination) to extend the construction to three dimensions, adding other controlling coils and allowing W freedom to turn about any axis in its own plane. This gives a model embodying the same principles, with all its coils in the form of orbits which have the nucleus of the atom at their focus,

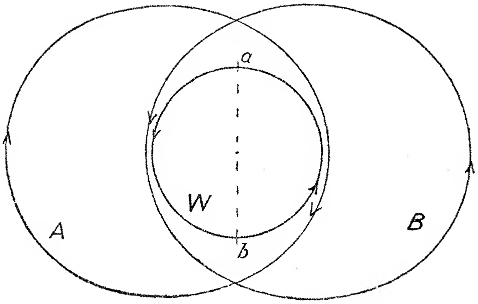

Fig 3.

With regard to the inequality of forces on which the control depends, there is an important distinction between (1) any inequality that may arise from the mutual action between the Weber element itself and the fixed elements; and (2) any inequality that may arise from strain in the piece as a whole, or from the influence of unsymmetrically placed neighbouring atoms. In case (1) the inequality is transferred from one to another pair of controlling elements as the Weber element turns from one position of stable equilibrium to another. Thus in the model (fig. 1), when W points towards A and B, there is inequality in the action of A and B, A being, by hypothesis, caused to act more strongly as an effect of the proximity to it of the unlike pole of W; but when W turns into another stable position between C and D, C for the same reason acts on it more strongly than D, and so on with any other pair. Except for the distorting influence of W, the atom may be isotropic in respect of the grouping of the elements A, B, C, D, etc., and the magnetic qualities of the piece will then be the same along these various axes. In case (2) on the other hand, there is an asymmetry which persists in particular directions within the atom while W turns, and its influence on the control is superposed on any that W itself produces, with the result that the piece may exhibit differences of susceptibility and retentiveness in different directions.

It is well known that a homogeneous ferromagnetic metal becomes magnetically æolotropic under strain; it is also well known that a compound crystal, such as pyrrhotite, has widely different magnetic qualities in different directions. I suggest that these are examples of case (2); they are effects which can be imitated in the model by setting up differences in the closeness of the paired controlling magnets or coils along different axes, and by making the turning element contain a sufficient number of magnet poles or coils.

It may be convenient to recapitulate the chief points of this paper. While retaining the idea of magnetic control of the Weber element as determining the susceptibility and hysteresis of ferromagnetic substances, I have been led to replace my older model by a new one in which the Weber element in each atom is controlled by other parts of the same atom instead of only by the Weber elements in the adjoining atoms. The position of the atom in the crystal secures that these parts are more or less completely fixed. They probably constitute a shell, or series of concentric shells, within which the Weber element may turn in response to an impressed field, and by which its turning is so controlled that it is capable of small stable deflections and of large irreversible deflections, so that it may swing over with dissipation of energy from one stable position to another when the range of stable deflection is exceeded. This range is in general very narrow. Experiments on the magnetic quality of soft iron show that it is less, in that metal, than 1°. At the same time they show that the control must be weak, for a small magnetising force suffices to upset the equilibrium of most of the Weber elements. The old model was defective because in it a narrow range of stable deflection could be secured only by placing the magnets so close together that their stability became far too great. Its quantitative failure in this respect was made apparent by considering the magnitude of the field required to break up pairs or rows of pivoted magnets. In the new model a narrow range of stable deflection is secured without excessive stability. This is because the model is so devised that the control of the Weber element in each atom depends on a slight inequality between strong opposing forces. These forces are exerted separately on different parts of it by other constituents of the atom. Examples of models satisfying this condition are described, with the equivalents of both large and small electron orbits, which reproduce the known characteristics of ferromagnetic induction.

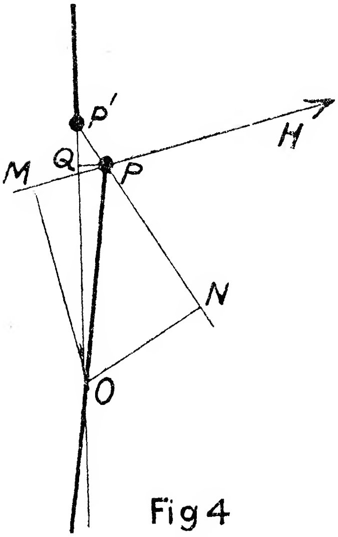

Fig 4

[*December* 14, 1921.—It is important to notice that the limit of stable deflection for a magnet pivoted as in fig. 1 between closely adjoining fixed poles is determined simply by the closeness of the poles, and is independent of their absolute or relative strengths. Let O, fig. 4, be the centre about which the pivoted magnet turns, P be one of its poles, and P′ be the adjoining fixed pole. Let m be the pole strength of P and let m' and m'' be the respective pole strengths of P′ and its opposite neighbour. Assume that the same geometrical conditions hold at both ends of the pivoted magnet. Suppose a field, H, to act, with a constant inclination α to the line of centres, OP′, with the result of deflecting the pivoted magnet through a small angle QOP or θ. Write r for OP, the half length of the pivoted magnet, a for OP′, x for PQ, and c for PP′. The deflecting moment due to the action of H on P is $m\text{H} \cdot \text{OM} = m\text{H}r\sin(\alpha-\theta)$. The restoring moment due to the action of P′ on P is

$$\frac{mm'}{c^2} \cdot \text{ON} = \frac{mm'ax}{c^3},$$

Equating deflecting and restoring moments, we have

$$2m\mathrm{H}r\sin(\alpha-\theta) = \frac{m(m'-m'')ax}{c^3}.$$

When H is increased the limit of stable deflection is found by writing

$$\frac{d}{dx}\cdot\frac{x}{c^3\sin(\alpha-\theta)} = 0.$$

We are concerned only with very small limiting values of θ and it will simplify the expression if we confine it to cases in which $\sin(\alpha-\theta)$ may be treated as sensibly equal to $\sin\alpha$, and therefore sensibly constant, namely cases in which θ is very small and α is not small. Then the criterion for the rupture of stability becomes

$$\frac{d}{dx}\left(\frac{x}{c^3}\right) = 0,$$

which gives $x = (a-r)/\sqrt{2}$, since $c^2 = x^2+(a-r)^2$. In other words the deflection ceases to be stable when the angle OP′P increases to $\tan^{-1} 1/\sqrt{2}$ or 35° 16′. This is a purely geometrical criterion: it is independent of the magnitudes of m and of the fixed poles. In the calculation we have assumed the distances of the fixed poles to be the same at both ends of the pivoted magnet, but the stability may be adjusted to any desired value by making the attracting fixed pole stronger than the repelling fixed pole. Thus a narrow range of stable deflection may be retained even when the control is made vanishingly small by having the fixed poles at both ends approach equality in strength as well as distance.

These considerations suggest a further point. Imagine a continuous row of model atoms with fixed magnets lying in one straight line, and in which the opposite fixed poles P′ and P″ are in fact equal and spaced at equal distances from the adjoining moving poles P and −P. Though in each atom of the row the action of one fixed pole on the pivoted magnet exactly balances that of the other, there will still be some control, for the pivoted magnets will act on one another as they acted in my model of 1890, but across the intervening gap formed by the outer portions of contiguous atoms. Imagine now a model crystal made up of such atoms. The pivoted magnets will tend to form rows in which the fixed magnets of the intervening gap will also be members. In any one row half the number of fixed magnets will point one way and half the other, but the pivoted magnets will all point one way. Owing to the relatively great distance that separates each pivoted magnet from its nearest pivoted neighbours, the control that is due to the mutual forces between the pivoted magnets will be very weak; but since there is (by hypothesis) a nice balance between the forces exerted by the opposing fixed poles, the mutual forces between the pivoted magnets become the operative factor in producing residual magnetism and the other phenomena of hysteresis. The action in the new model would then resemble that of my model of 1890, but with this fundamental difference, that in the new model the mutual forces between the pivoted magnets are exerted across a wide gap, and in that gap there are fixed magnets whose effect is to make the range of stable deflection very narrow, although they do not themselves provide the control which determines the direction the pivoted magnets will assume.

It is at least open to conjecture that in an ideal crystal of perfectly pure and perfectly unstrained iron, what I have called the fixed magnets in each atom may be completely symmetrical and rigid in maintaining their symmetry, while the Weber element turns. In that case the mutual directing forces exerted by the Weber elements on one another, from atom to atom, might still provide the small amount of stability that is required when superposed on the far stronger but balanced forces which are due to the fixed magnets. When, however, the iron is impure, or is rendered æolotropic by straining, it is natural to suppose that each atom is more or less distorted; in any case the balance of magnetic forces within it is disturbed, and a stronger type of control, due to unequal action on the part of the fixed magnets, is brought into play.]